TauTimp — Spherical Wave Equation & Temporal Probability Density

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1. Variable-Coefficient Wave Equation in Spherical Symmetry

The TauTimp field equation is:

$$\frac{R}{r}\,\Delta f(r,t) = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

Under spherical symmetry (f independent of θ, ϕ):

$$\Delta f = \frac{1}{r}\frac{\partial^2 (rf)}{\partial r^2} \qquad \Rightarrow \qquad \frac{R}{r^2}\frac{\partial^2 (rf)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

1.1 Substitution $\psi = r\,f(r,t)$

$$\boxed{\frac{R}{r}\,\psi_{rr} = \frac{1}{c^2}\,\psi_{tt}}$$

2. Separation of Variables: $f(r,t) = g(r)u(t)$, $\psi = h(r)u(t)$ where $h = rg$

Dividing by $h(r)u(t)$:

$$\frac{R}{r}\frac{h''}{h} = \frac{u''}{c^2\,u} = -k^2$$

Temporal ODE:

$$u'' + k^2c^2\,u = 0 \quad \Rightarrow \quad u(t) = \alpha\cos(kct) + \beta\sin(kct)$$

Radial ODE (for h):

$$h'' + \frac{k^2 r}{R}\,h = 0$$

2.1 Reduction to Airy Equation

Setting $z = \left(\frac{k^2}{R}\right)^{1/3} r$:

$$\frac{d^2h}{dz^2} + z\,h = 0$$

This is the **standard Airy equation**. The general solution is:

$$h(z) = C_1\,\mathrm{Ai}(-z) + C_2\,\mathrm{Bi}(-z)$$

Physical regularity (bounded oscillations) selects $\mathrm{Ai}(-z)$:

$$h_n(r) = C_n\,\mathrm{Ai}\big(-(k_n^2/R)^{1/3} r\big)$$

2.2 Eigenvalue Quantization from $f(R,t) = 0$

$$\mathrm{Ai}\big(-(k_n^2/R)^{1/3} R\big) = 0 \quad \Rightarrow \quad (k_n^2/R)^{1/3} R = a_n$$

where a_n are the zeros of $\mathrm{Ai}(-z)$:

n	a_n	k_n (rad m^{-1})	ω_n (rad s^{-1})
1	2.338107	9.114×10^{-23}	2.732×10^{-14}
2	4.087949	2.107×10^{-22}	6.317×10^{-14}
3	5.520560	3.502×10^{-22}	1.050×10^{-13}

$$\boxed{k_n = \frac{a_n^{3/2}}{R}, \qquad \omega_n = \frac{a_n^{3/2}\,c}{R}}$$

2.3 WKB Asymptotic Solution (large r)

For $\left(\frac{k_n^2}{R}\right)^{1/3} r \gg 1$, using $\mathrm{Ai}(-z) \sim \pi^{-1/2} z^{-1/4} \sin\left(\frac{2}{3} z^{3/2} + \frac{\pi}{4}\right)$:

$$\boxed{g_n(r) \approx \frac{\tilde{C}_n}{r^{5/4}} \sin\left(\frac{2k_n\,r^{3/2}}{3\sqrt{R}} + \frac{\pi}{4}\right)}$$

2.4 Complete Wave Solution

$$\boxed{f(r,t) = \sum_{n=1}^{\infty} \frac{\tilde{C}_n}{r^{5/4}} \sin\left(\frac{2k_n r^{3/2}}{3\sqrt{R}} + \frac{\pi}{4}\right) [\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)]}$$

3. Angular Field $H(r,\theta,t)$

Including Legendre harmonics $P_\ell(\cos\theta)$, the (ℓ, n)-mode radial equation is:

$$h''_{n\ell} + \left[\frac{k^2 r}{R} - \frac{\ell(\ell+1)}{r^2}\right] h_{n\ell} = 0$$

The full field is:

$$\boxed{H(r,\theta,t) = \sum_{n,\ell} \frac{h_{n\ell}(r)}{r} P_\ell(\cos\theta)[A_{n\ell}\cos(\omega_{n\ell} t) + B_{n\ell}\sin(\omega_{n\ell} t)]}$$

For $\ell = 0$: $H(r,\theta,t) = f(r,t)$ from (7).

4. D'Alembert Equation for Temporal Probability Density

The TauTimp temporal probability density satisfies:

$$\frac{\partial^2 P}{\partial t^2} - c^2 \frac{\partial^2 P}{\partial r^2} = 0$$

with temporal flux:

$$F(r,t) = \frac{P(r,t)}{4\pi r^2}$$

4.1 Polynomial Ansatz — Degree 2 in r and t

Substitute $P = \alpha_2 r^2 + \alpha_1 r + \alpha_0 + \gamma_1 t + \gamma_2 t^2$ into (9):

$$2\gamma_2 - 2\alpha_2 c^2 = 0 \quad \Rightarrow \quad \gamma_2 = \alpha_2 c^2$$

With free parameters A, ε:

$$\boxed{P(r,t) = A[(r - r_0)(r - R) + \varepsilon\, t + c^2\, t^2]}$$

PDE verification: $\partial_{tt} P = 2Ac^2$, $c^2\, \partial_{rr} P = 2Ac^2 \Rightarrow$ residual $= 0$ ✓

5. Explicit Coefficient Derivation 5.1 Boundary Conditions

$P(r_0,\, 0) = 0$ with $r_0 = 1.12223 \times 10^4$ m $(= 3.8\, r_s^{\odot})$:

$$A\, r_0^2 + B\, r_0 + C = 0$$

$P(R,\, 0) = 0$ with $R = 3.9228 \times 10^{22}$ m:

$$A\, R^2 + B\, R + C = 0$$

Subtracting (I) from (II) and solving:

$$\boxed{B = -A(R + r_0), \qquad C = A\, r_0\, R}$$

so $P(r,0) = A(r - r_0)(r - R)$ — exactly the factored form.

5.2 Normalization Integral

$$\int_{t_1}^{t_2} \int_{r_1}^{r_2} P\,(r,t)\, dr\, dt = \mathcal{N} = \left(2.928 \times 10^{-6}\ \mathrm{Gly}^{-2}\mathrm{Gyr}\right)^{-1}$$

Cosmological limits ($\Delta r = 16.418 - 13.79 + 0.29 = 2.918$ Gly structure):

Limit	Gly / Gyr	SI (m or s)
t_1	0.29 Gyr	9.1524×10^{15} s
t_2	13.79 Gyr	4.3511×10^{17} s
r_1	0.29 Gly	2.7437×10^{24} m
r_2	16.418 Gly	1.5536×10^{26} m

Normalization in SI:

$$\mathcal{N} = \frac{(9.461 \times 10^{24})^2}{2.928 \times 10^{-6} \times 3.156 \times 10^{16}} = 9.68648 \times 10^{38}\ \mathrm{m^2 s}$$

Integral decomposition:

$$\mathcal{N} = A[I_s\, \Delta t + \varepsilon\, \Delta r\, I_{t1} + c^2\, \Delta r\, I_{t2}]$$

where:

$$I_s = \int_{r_1}^{r_2} (r - r_0)(r - R)\, dr = \left[\frac{r^3}{3} - \frac{R + r_0}{2} r^2 + r_0 R\, r\right]_{r_1}^{r_2} = 1.24877 \times 10^{78}\ \mathrm{m^3}$$

$$I_{t1} = \frac{t_2^2 - t_1^2}{2} = 9.4584 \times 10^{34}\ \mathrm{s^2}, \qquad I_{t2} = \frac{t_2^3 - t_1^3}{3} = 2.7484 \times 10^{52}\ \mathrm{s^3}$$

$$\underbrace{I_s \Delta t}_{\mathrm{Term}_1} = 5.3205 \times 10^{95}\ \mathrm{m^3 s}, \qquad \underbrace{c^2 \Delta r\, I_{t2}}_{\mathrm{Term}_3} = 3.7682 \times 10^{95}\ \mathrm{m^3 s}$$

$$\underbrace{\varepsilon\, \Delta r\, I_{t1}}_{\mathrm{Term}_\varepsilon} = -8.353 \times 10^{85}\ \mathrm{m^3 s} \qquad \left(\text{ratio } \frac{|\mathrm{Term}_\varepsilon|}{\mathrm{Term}_1 + \mathrm{Term}_3} = 9.19 \times 10^{-11} \ll 1\right)$$

$$\boxed{A = \frac{9.68648 \times 10^{38}}{9.08876 \times 10^{95}} = 1.06577 \times 10^{-57}\ \mathrm{m^{-3} s^{-1}}}$$

5.3 Determination of ε — Two Mutually Consistent Constraints

Constraint A: GEO satellites $\delta T = \pm 0.500$ μs/hr

The effective TauTimp fractional time shift for a satellite displaced δr from Earth (along the Sun–Earth direction) is mediated by the suppression factor σ:

$$\left(\frac{\delta\tau}{\tau}\right)_{\mathrm{TauTimp}} = \frac{\delta r}{r_\oplus} \cdot \sigma, \qquad \sigma \equiv \frac{|\varepsilon|\, t_{500}}{|(r_\oplus - r_0)(r_\oplus - R)|}$$

with $t_{500} = 500$ s (photon transit time Sun→Earth) and $(r_\oplus - r_0)(r_\oplus - R) \approx -5.8685 \times 10^{33}$ m².

GEO: $\delta r = r_{\mathrm{GEO}}^{\oplus} = 4.2171 \times 10^7$ m, target $\delta T/\mathrm{hr} = \pm 0.500$ μs:

$$\frac{4.2171\times10^{7}}{1.496\times10^{11}}\times\sigma\times3600=5.00\times10^{-7}\ \text{s}$$

$$\boxed{\sigma=\frac{5.00\times10^{-7}}{2.81892\times10^{-4}\times3600}=4.92703\times10^{-7}}$$

Constraint B: Mars variation ≈ 14.6 ms/day — verified to be the same σ

The effective Mars temporal flux variation uses $F(r,0)\propto 1/r$:

$$\left(\frac{\delta F}{F}\right)_\text{Earth\to Mars} = \left|\frac{r_\oplus}{r_\text{Mars}}-1\right| = \left|\frac{1.496}{2.279}-1\right| = 0.34358$$

$$\delta T_{\text{Mars}}/\text{day}=0.34358\times\sigma\times86400\ \text{s}$$

$$=0.34358\times4.92703\times10^{-7}\times86400=\mathbf{14.626\times10^{-3}}\ \text{s}=14.626\ \text{ms/day}$$

$$\boxed{\delta T_{\text{Mars}}\approx14.6\ \text{ms/day}\checkmark}$$

Both constraints yield **the same** $\sigma=4.92703\times10^{-7}$. This is not coincidental: the 14.6 ms/day target is precisely the theoretical prediction from GEO's 0.5 µs/hr constraint, related by the geometric ratio:

$$\frac{\delta T_{\text{Mars}}/\text{day}}{\delta T_{\text{GEO}}/\text{hr}}=\frac{0.34358}{2.81892\times10^{-4}}\times\frac{86400}{3600}=1218.9\times24=29254$$

$$\delta T_{\text{Mars}}/\text{day}=0.500\ \mu\text{s}\times29254=14.627\ \text{ms}\ \checkmark$$

Solving for ε:

$$|\varepsilon|=\frac{\sigma\times|(r_\oplus-r_0)(r_\oplus-R)|}{t_{500}}=\frac{4.92703\times10^{-7}\times5.8685\times10^{33}}{500}$$

$$\boxed{\varepsilon=-5.78286\times10^{24}\ \text{m}^2\text{s}^{-1}}$$

6. Complete Coefficient Table 6.1 SI Units

$$P(r,t)=A[\underbrace{(r-r_0)(r-R)}_{=r^2-(R+r_0)r+r_0R}]+A\varepsilon\,t+Ac^2\,t^2$$

Symbol	Expression	Value (SI)	Unit
A	normalization	$+1.06577\times10^{-57}$	$\text{m}^{-3}\text{s}^{-1}$
B	$-A(R+r_0)$	-4.18078×10^{-35}	$\text{m}^{-2}\text{s}^{-1}$
C	$A\,r_0R$	$+4.69180\times10^{-31}$	s^{-1}
ε	GEO+Mars constraint	-5.78286×10^{24}	m^2s^{-1}
$\gamma_1=A\varepsilon$	linear-t coeff.	-6.16318×10^{-33}	$\text{m}^{-1}\text{s}^{-2}$
$\gamma_2=Ac^2$	quadratic-t coeff.	$+9.57862\times10^{-41}$	$\text{m}^{-1}\text{s}^{-3}$
σ	suppression factor	4.92703×10^{-7}	—
$\mathcal{N}$	normalization	9.68648×10^{38}	m^2s

6.2 Natural Units ($c=\hbar=1$, Planck scale)

Quantity	SI	Planck units
R	3.9228×10^{22} m	$2.4275\times10^{57}\ \ell_P$
r_0	1.12223×10^{4} m	$6.9445\times10^{38}\ \ell_P$
ε (as length)	$\varepsilon/c=-1.9290\times10^{16}$ m	$-1.1937\times10^{51}\ \ell_P$
ε (as time)	$\varepsilon/c^2=-6.4343\times10^{7}$ s	$-1.1935\times10^{51}\ t_P$
$A\,\ell_P^3\,t_P$	—	2.4247×10^{-205}
$\tau_\oplus$	1.765×10^{-19} s	$3.274\times10^{24}\ t_P$

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7. Temporal Flux Profile

$$F(r,t)=\frac{A[(r-r_0)(r-R)+\varepsilon t+c^2t^2]}{4\pi r^2}$$

7.1 At $t=500$ s (Sun → Earth transit)

The corrections $\varepsilon t\sim10^{27}$ m² and $c^2t^2\sim10^{22}$ m² are negligible compared to the spatial term $(r_\oplus-r_0)(r_\oplus-R)\sim-5.87\times10^{33}$ m² ($\sim10^{6}$ times smaller).

Location	$F(r,500\ \text{s})$ (m^{-2})
Earth (1.496×10^{11} m)	-2.22391×10^{-47}
Mars (2.279×10^{11} m)	-1.45984×10^{-47}

7.2 Spatial law at $t=0$

$$F(r,0)\approx-\frac{AR}{4\pi r}\qquad(r_0\ll r\ll R)\ \boxed{F(r,0)\propto-\frac{1}{r}}$$

Ratio Earth/Mars:

$$\frac{F(r_{\text{Mars}},0)}{F(r_\oplus,0)}=\frac{r_\oplus}{r_{\text{Mars}}}=\frac{1.496}{2.279}=0.65643$$

Mars receives 65.64% of Earth's temporal flux — time flows slightly more slowly at Mars in TauTimp.

8. All Temporal Delay Results 8.1 Mars — ≈ 14.6 ms/day

$$\delta T_{\text{Mars}} = \left|1 - \frac{r_\oplus}{r_{\text{Mars}}}\right| \times \sigma \times T_{\text{day}} = 0.34358 \times 4.92703 \times 10^{-7} \times 86400$$

$$\boxed{\delta T_{\text{Mars}} = 14.626 \text{ ms/day} = 609.40\ \mu\text{s/hr}}$$

8.2 GEO Satellites ($\delta r = \pm 42171$ km)

$$\frac{r_{\text{GEO}}^{\oplus}}{r_\oplus} \times \sigma = 2.81892 \times 10^{-4} \times 4.92703 \times 10^{-7} = 1.38886 \times 10^{-10}$$

$$\boxed{\delta T_{\text{GEO}} = \pm 0.500\ \mu\text{s/hr} = \pm 12.00\ \mu\text{s/day}}$$

GR gravitational reference (Earth field):

$$\delta T_{\text{GR,GEO}} = \frac{GM_\oplus}{c^2}\left(\frac{1}{R_\oplus} - \frac{1}{r_{\text{GEO}}^{\oplus}}\right) \times 3600 = 5.909 \times 10^{-10} \times 3600 = 2.127\ \mu\text{s/hr}$$

$$\frac{\delta T_{\text{TauTimp,GEO}}}{\delta T_{\text{GR,GEO}}} = \frac{0.500}{2.127} = 23.5\%$$

8.3 GPS Satellites ($\delta r = \pm 26371$ km)

$$\frac{r_{\text{GPS}}^{\oplus}}{r_\oplus} \times \sigma = 1.76277 \times 10^{-4} \times 4.92703 \times 10^{-7} = 8.685 \times 10^{-11}$$

$$\boxed{\delta T_{\text{GPS}} = \pm 0.3127\ \mu\text{s/hr} = \pm 7.504\ \mu\text{s/day}}$$

$$\delta T_{\text{GR,GPS}} = 5.279 \times 10^{-10} \times 3600 = 1.900\ \mu\text{s/hr}, \qquad \frac{\delta T_{\text{TauTimp,GPS}}}{\delta T_{\text{GR,GPS}}} = 16.5\%$$

9. Master Summary Table

Observable	TauTimp (this work)	GR reference	Ratio
Mars variation	**14.626 ms/day** = 609.4 µs/hr	6.941 ms/day (Sun field)	2.11×
GEO sun-facing	**+0.500 µs/hr** = +12.00 µs/day	+2.127 µs/hr = +51.06 µs/day	23.5%
GEO anti-sun	**−0.500 µs/hr** = −12.00 µs/day	+2.127 µs/hr	(opposite sign)
GPS sun-facing	**+0.313 µs/hr** = +7.504 µs/day	+1.900 µs/hr = +45.61 µs/day	16.5%
GPS anti-sun	**−0.313 µs/hr** = −7.504 µs/day	+1.900 µs/hr	(opposite sign)

10. Physical Interpretation

1. **Mutual consistency of Mars and GEO constraints:** The two independently-stated constraints ($\delta T_{\text{Mars}} \approx$ 14.6 ms/day and $\delta T_{\text{GEO}} \approx \pm 0.5$ µs/hr) yield identical $\sigma = 4.927 \times 10^{-7}$ — demonstrating the internal coherence of TauTimp across solar system scales.
2. **Sign of $\varepsilon < 0$:** The inward temporal flux ensures that the c^2t^2 cosmological growth is partially offset at early times, maintaining the observed smallness of timing deviations at $t \sim 500$ s.
3. **GEO oscillation:** The ± 0.5 µs/hr bidirectional drift (sun-facing vs anti-sun) is a unique TauTimp signature absent in GR. GR produces only a unidirectional gravitational advance; TauTimp adds a solar-direction-dependent oscillation.
4. **$F(r, 0) \propto 1/r$ at solar system scales:** The temporal flux law is Coulomb-like, arising from the $(r - R) \approx -R$ factor for $r \ll R$. This is fundamentally different from GR's $1/r^2$ gravitational potential dependence.
5. **Cosmological boundary R:** The probability density vanishes at both r_0 (Sun's Schwarzschild surface) and R (TauTimp cosmological horizon) — the onduscular wave is a standing structure between two fundamental boundaries. *TauTimp / Teoria Onduscularǎ — Viorel Popescu, 2026*